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APPROXIMATE RIEMANN SOLVERS AND NUMERICAL FLUX FUNCTIONS

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Abstract

Given a monotone function $z(x)$ which connects two constant states, $u_L < u_R$, $(u_L > u_R)$, we find the unique (up to a constant) convex (concave) flux function, $\hat{f}(u)$, such that $z(x/t)$ is the physically correct solution to the associated Riemann problem. For $z(x/t)$, an approximate Riemann solver to a given conservation law, we derive simple necessary and sufficient conditions for it to be consistent with any entropy inequality. Associated with any member of a general class of consistent numerical fluxes, $h_f(u_R,u_L)$, we have an approximate Riemann solver defined through $z(\zeta) = (-d/d_\zeta)h_{f_\zeta}(u_R,u_L)$, where $f_\zeta(u) = f(u) - \zeta u$. We obtain the corresponding $\hat{f}(u)$ via a Legendre transform and show that it is consistent with all entropy inequalities iff $h_{f_\zeta}(u_R,u_L)$ is an E flux for each relevant ζ. Examples involving commonly used two point numerical fluxes are given, as are comparisons with related work.

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0. *Introduction*

The aim of this paper is to identify an approximate solution to the Riemann problem for hyperbolic systems of conservation laws with any of a wide class of three point shock capturing algorithms used to approximate these laws. We also wish to analyze consistency of this approximation with any analytic entropy condition. This procedure will be done here only for scalar conservation laws; systems will be considered in the future.

We begin with an arbitrary monotone function, $z(x)$, connecting two constant states $u_L < u_R$, or $u_L > u_R$; we find the unique (up to a constant) convex (concave) function, $\hat{f}(u)$, such that $z(\frac{x}{t})$ is the solution to the associated Riemann problem

(0.1(a) $$u_t + \hat{f}(u)_x = 0, \quad t > 0$$

(0.1)(b) $$\begin{aligned} u(x,0) &\equiv u_L, \quad x \le 0 \\ u(x,0) &\equiv u_R, \quad x > 0 \end{aligned}$$

which satisfies Kruz'kov's entropy condition [7], discussed in the next section. The resulting formula involves a Legendre transform.

For an arbitrary flux function $f(u)$, we define an approximate Riemann solver to

(0.2)(a) $$u_t + f(u)_x = 0$$

(0.2)(b) $$\begin{aligned} u(x,0) &\equiv u_L, \quad x \le 0 \\ u(x,0) &\equiv u_R, \quad x > 0, \end{aligned}$$

of conservation with (0.2). Under this circumstance we find a simple geometric criterion, relating the graphs of f and $\hat{f}$, for such an approximation to be consistent with any entropy condition.

Next, we introduce an approximate Riemann solver associated with any consistent numerical flux function if h_f satisfies a fairly nonrestrictive condition, concerning its numerical viscosity. The corresponding $\hat{f}(u)$ is obtained through a Legendre transform of the function $h_{f_\zeta}(u_R,u_L)$, with respect to ζ.

Here we define

$$f_\zeta(u) = f(u) - \zeta u. \tag{0.3}$$

This function of ζ is obtained by constructing the numerical flux on a grid moving with speed $x = \zeta t$. We show that $z(\zeta)$ is consistent with Kruz'kov's entropy condition, [7], iff $h_{f_\zeta}(u_R,u_L)$ is an E flux (introduced in [11]).

We then illustrate our theory by presenting examples involving all the commonly used two point numerical flux functions.

We conclude by comparing our approach to that of Harten and Lax [5], and Harten, Lax, and van Leer [6], in the scalar case. (They also discussed systems.)

We are motivated to study these problems because of their connection with the classic work of Glimm [3]. In that paper, he obtained a celebrated existence theorem for a class of hyperbolic systems of conservation laws whose initial data differs slightly from a constant state. His method of proof was constructive. He represented the approximate solutions as piecewise constant at any time; they were advanced in time by solving exactly the Riemann problem formed by the constant states between two neighboring cells. The value of the approximation in each cell at the new time was taken to be the value of the exact solution at a randomly chosen point in the cell. Glimm then proved the convergence of this method.

Recently, the first author [1] modified Glimm's approach by replacing the exact solution to the Riemann problem between two neighboring cells by the solution to the associated linear Riemann problem constructed by Roe [13]. This linear Riemann problem changes from cell to cell, and its solution is consistent, in the sense of conservation, with each of the true Riemann problems. Following, but somewhat simplifying, Glimm's technique, for initial data of his type, a variation estimate is obtained for these approximate solutions. Using a theorem of Harten-Lax [5], it follows that a new existence theorem for systems of equations which may have fields which are neither linearly degenerate or genuinely nonlinear, as defined in [8], is obtained. Unfortunately, these limit solutions will not, in general, satisfy the entropy condition of Lax [8]. Work is under way to remedy this difficulty.

Roe's approximate Riemann solver was constructed as a step in the generation of a numerical flux for a conservation form, shock capturing method. Nevertheless, it has proven to be useful theoretically in a random choice setting. This fact motivates the present work.

I. *The Riemann Problem and Approximations*

We shall consider approximations to the initial value problem for a scalar hyperbolic equation

$$\text{(1.1)(a)} \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad t > 0, \quad -\infty < x < \infty$$

with special initial data

$$\text{(1.1)(b)} \qquad \begin{aligned} u(x,0) &\equiv u_L, \quad x < 0 \\ u(x,0) &\equiv u_R, \quad x > 0 \end{aligned}$$

for arbitrary constants, u_L, u_R.

Solutions to the Riemann problem (1.1) are not unique. For physical reasons, the limit solution of the viscous equation, as viscosity tends to zero, is sought. This solution must satisfy the entropy inequality for all real c:

$$\text{(1.2)} \qquad \frac{\partial}{\partial t}(u - c)_+ + \frac{\partial}{\partial x}(f(u) - f(c))\chi(u - c)) \leq 0$$

in the sense of distributions.

Here:

$$\text{(1.3)} \qquad \begin{aligned} (a)_+ &= \max(a,0) \\ \chi(a) &= 1 \text{ if } a \geq 0 \\ \chi(a) &= 0 \text{ if } a < 0 \end{aligned}$$

This solution to (1.1), (1.2) is unique -- see e.g. Kruz'kov [7].

If f is strictly convex or concave, the solution is particularly simple, and may be characterized by replacing (1.2) with a single entropy inequality, rather than the infinite family. If $V(u)$ is any strictly convex function of a (1.2) may then be replaced by

(1.4)(a) $$\frac{\partial}{\partial t} V(u) + \frac{\partial}{\partial x} F(u) \le 0$$

where

(1.4)(b) $$F'(u) = V'(u)f'(u).$$

Inequalities (1.2) have important consequences for piecewise continuous solutions. Suppose $u(x,t)$ is such a solution having a jump discontinuity $u_L(t)$, $u_R(t)$, moving with speed $s(t)$. Then (1.2) implies the well known jump conditions

(1.5) $$f(u_L) - f(u_R) = s(u_L - u_R)$$

and Oleinik's condition E across the shock:

(1.6) $$\frac{f(u) - f(u_R)}{u - u_R} \le \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

for all u between u_L and u_R. We call the corresponding interval I_{RL}

$$I_{RL} = \{u \mid \min[u_L, u_R] \le u \le \max[u_L, u_R]\}$$

If f is convex, then (1.6) is equivalent to the statement that characteristics flow *into* the discontinuity as time increases.

In [1], the second author obtained the following formula for the solution to (1.1), (1.2). Let $\zeta = x/t$. The solution is a function of ζ defined by:

THEOREM (1.1)

(1.7)(a) *If* $u_L < u_R$, *then*

$$u(\zeta) = \frac{d}{d\zeta}\left(\max_{u \in [u_L,u_R]} [\zeta u - f(u)]\right)$$

(1.7)(b) *If* $u_L > u_R$, *then*

$$u(\zeta) = \frac{d}{d\zeta}\left(\min_{u \in [u_R,u_L]} [\zeta u - f(u)]\right).$$

This result follows directly from the following:

LEMMA (1.1)

(1.8)(a) *If* $u_L < u_R$, *then*

$$\zeta u(\zeta) - f(u(\zeta)) = \max_{u \in [u_L,u_R]} [\zeta u - f(u)].$$

(1.8)(b) *If* $u_L > u_R$, *then*

$$\zeta u(\zeta) - f(u(\zeta)) = \min_{u \in [u_R,u_L]} [\zeta u - f(u)].$$

It is easy to see that each expression on the right above is Lipschitz with Lipschitz constant $\max(|u_L|, |u_R|)$, and is a convex (concave) function of ζ.

Also

$$u(\zeta) \equiv u_L, \text{ if } \zeta < \min_{u \in I_{RL}} f'(u)$$

$$u(\zeta) \equiv u_R, \text{ if } \zeta > \max_{u \in I_{RL}} f'(u).$$

Motivated by this, we consider a function, $z(\zeta)$, having the following properties:

(a) $z(\zeta) \equiv u_L, \quad -\infty < \zeta \leq \zeta_L$

(b) $z(\zeta) \equiv u_R, \quad \zeta_R \leq \zeta \leq \infty$

(c) $z(\zeta)$ is monotone.

We can now state our first new result:

THEOREM (1.2). $z(\zeta) = z(x/t)$ *is the entropy condition satisfying solution to the Riemann problem:*

$$
\begin{aligned}
(1.8) \qquad & z_t + \hat{f}(z)_x = 0 \\
& z(x,0) \equiv u_L, \quad x < 0 \\
& z(x,0) \equiv u_R, \quad x > 0
\end{aligned}
$$

where:

$$(1.9)(a)\ \hat{f}(u) = \hat{f}(u_L) - \zeta_L u_L + \max_{\zeta_L \leq \zeta \leq \zeta_R} [\zeta u - \int_{\zeta_L}^{\zeta} z(s)\, ds], \quad \text{if } u_L \leq u_R$$

$$(1.9)(b)\ \hat{f}(u) = \hat{f}(u_L) - \zeta_L u_L + \min_{\zeta_L \leq \zeta \leq \zeta_R} [\zeta u - \int_{\zeta_L}^{\zeta} z(s)\, ds], \quad \text{if } u_L > u_R$$

The function $\hat{f}(u)$ *is unique up to an additive constant among convex (concave) flux functions.*

Proof. We use Theorem (1.1) and equation (1.7)(a), if $u_L \leq u_R$. Integrating from ζ_L to ζ, gives us:

$$\zeta_L u_L - \hat{f}(u_L) + \int_{\zeta_L}^{\zeta} z(s)\, ds = \max_{u \in I_{RL}} [\zeta u - \hat{f}(u)]$$

We then take the Legendre transform and obtain uniqueness, given that $\hat{f}(u)$ is convex.

For existence, we note that $-\int_{\zeta_L}^{\zeta} z(s)\, ds$ is convex if $u_L \le u_R$; thus the Legendre transform of $\hat{f}(u)$ in (1.8a) is $\int_{\zeta_L}^{\zeta} z(s)\, ds$. Differentiating this serves to verify that $z(\zeta)$ is the solution to the Riemann problem.

The proof is similar for $u_L > u_R$.

Remark (1.1). If $z(\zeta)$ is *strictly* monotone when it takes on values between u_L and u_R, then $\hat{f}$ is strictly convex (concave) and $z^{-1}(u)$ exists. In fact

$$\hat{f}(u) = \int_{u_L}^{u} z^{-1}(s)\, ds + \hat{f}(u_L). \tag{1.11}$$

Here $z(\zeta)$ is a single rarefaction wave.

Remark (1.2). If $\hat{f}(u)$ is not strictly convex or concave, then $z(\zeta)$ might be singular, and conversely. For example, if

(a) $$z(\zeta) = (\zeta)^{\alpha+1}\mathrm{sgn}\,\zeta, \quad \text{for } 0 > \alpha > -1, \quad -1 \le \zeta \le 1.$$

(b) $$z(\zeta) = \mathrm{sgn}\,\zeta, \quad |\zeta| > 1.$$

Then

(c) $$\hat{f}(u) = \frac{\alpha+1}{\alpha+2}\,|u|^{\frac{\alpha+2}{\alpha+1}} + \hat{f}(-1) - \left(\frac{\alpha+1}{\alpha+2}\right)$$

Next we define approximate solutions to (1.1).

DEFINITION (1.1). A function $z(\zeta)$ having properties (1.8) is an approximate solution to the Riemann problem (1.1) if it satisfies the conservation relation:

$$(1.12) \qquad \int_{\zeta_L}^{\zeta_R} z(\zeta)\, d\zeta = \zeta_R u_R - \zeta_L u_L - [f(u_R) - f(u_L)]$$

where ζ_R, ζ_L are arbitrary constants such that:

$$\zeta_R \geq \max_{u \in I_{RL}} f'(u)$$

$$\zeta_L \leq \min_{u \in I_{RL}} f'(u).$$

We have proven that the associated $\hat{f}(u)$ is defined by

$$(1.13) \quad \hat{f}(u) = f(u_L) - \zeta_L u_L + \max_{\zeta_L \leq \zeta \leq \zeta_R} [\zeta u - \int_{\zeta_L}^{\zeta} z(s)\, ds], \quad \text{if } u_L < u_R$$

$$\hat{f}(u) = f(u_L) - \zeta_L u_L + \min_{\zeta_L \leq \zeta \leq \zeta_R} [\zeta u - \int_{\zeta_L}^{\zeta} z(s)\, ds), \quad \text{if } u_L > u_R$$

and

$$\hat{f}(u_L) = f(u_L)$$

$$\hat{f}(u_R) = f(u_R).$$

(The last two equalities follow from (1.12) and (1.13)).

We next consider the entropy inequality (1.4) associated with (1.1). As in the previous Lemma, we integrate (1.4) over the box

$$\Omega = [(x,t)/ - T\zeta_L \leq x \leq T\zeta_R, \quad 0 \leq t \leq T]$$

arriving at (after division by T):

$$(1.14) \quad \int_{\zeta_L}^{\zeta_R} V(u(\zeta))\, d\zeta \leq \zeta_R V(u_R) - \zeta_L V(u_L) - [F(u_R) - F(u_L)]$$

DEFINITION (1.2). We say the Riemann solver, $z(\zeta)$, approximating (1.1) is consistent with a given entropy inequality, (1.4), if

$$(1.15) \quad \int_{\zeta_L}^{\zeta_R} V(z(\zeta))\, d\zeta \leq \zeta_R V(u_R) - \zeta_L V(u_L) - [F(u_R) - F(u_L)].$$

We define it to be consistent with all entropy inequalities if (1.15) is true for $V(u) = (u - c)_+$, for any constant c.

We have the following:

THEOREM (1.3). The approximate Riemann solver $z(\zeta)$ is consistent with a given entropy inequality (1.4), iff the inequality

$$(1.16) \qquad \int_{u_L}^{u_R} V''(u)\, (\hat{f}(u) - f(u))\, du \leq 0$$

is valid; thus it is consistent with all entropy inequalities iff

$$(1.17) \qquad \operatorname{sgn}(u_R - u_L)\, \hat{f}(u) \leq \operatorname{sgn}(u_R - u_L)\, f(u) \quad \text{for } all \;\; u \in I_{RL}$$

Remark (1.3). The inequality (1.16) is a Riemann solver version of a discrete entropy-in-cell formula obtained by the second author for systems of equations in [11] - equation (3.6). A similar result was used later in [12].

Proof of Theorem (1.3).

Inequality (1.17) easily follows if (1.16) is valid, for any convex $V(u)$. It is, however, instructive to prove it directly. By Theorem (1.2), and Lemma (1.1) of [11], we have for any $a \in I_{RL}$.

$$z(\zeta) - a = \frac{d}{d\zeta}\, [\zeta(z(\zeta) - a) - (\hat{f}(z(\zeta)) - \hat{f}(a))].$$

We insert this in (1.15) for $V(z) = (z - a)_+$. We see that (1.15) is valid iff:

$$\operatorname{sgn}(u_R - u_L)[\zeta_R(u_R - a) - (f(u_R) - \hat{f}(a))]$$

$$\leq \operatorname{sgn}(u_R - u_L)[\zeta_R(u_R - a) - (f(u_R) - f(a))]$$

which is equivalent to (1.17). To prove (1.16), we have

$$\int_{\zeta_L}^{\zeta_R} V(z(\zeta))\, d\zeta = \int_{\zeta_L}^{\zeta_R} \left(\frac{d}{d\zeta}(\zeta)\right) V(z)\zeta))\, d\zeta$$

$$= \zeta_R V(u_R) - \zeta_L V(u_L) - \int_{\zeta_L}^{\zeta_R} z'(\zeta) V'(z(\zeta))\hat{f}'(z(\zeta)) d_\zeta$$

$$= \zeta_R V(u_R) - \zeta_L V(u_L) - \int_{u_L}^{u_R} V'(u)\hat{f}'(u)\, du,$$

where we use the fact that, by (1.16) of [11],

$$\zeta z'(\zeta) = z'(\zeta)\hat{f}'(z(\zeta)).$$

Thus (1.15) is valid iff:

$$0 \geq F(u_R) - F(u_L) - \int_{u_L}^{u_R} V'(u)\hat{f}'(u)\, du$$

$$= \int_{u_L}^{u_R} V'(u)(f'(u) - \hat{f}'(u))\, du = \int_{u_L}^{u_R} V''(u)\, (\hat{f}(u) - f(u)) du$$

II. *Numerical Fluxes and "E" Schemes.*

We can approximate solution to (1.1a) with general initial data via a conservation form discretization.

$$(2.1) \quad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(h_f(u_{j+1}^n, u_j^n) - h_f(u_j^n, u_{j-1}^n)\right)$$

$$= u_j^n - \frac{\Delta t}{\Delta x}\Delta_- h_f(u_{j+1}^n, u_j^n).$$

Here $u_\Delta(x,t)$ is a piecewise constant approximation to $u(x,t)$, defined through

$$u_\Delta(x,t) \equiv u_j^n$$

for

$$(x,t) \in I_j \times [t^n, t^n + \Delta t], \quad t^n = n\,\Delta t, \quad n = 0,1,\ldots.$$

$$I_j = (x \mid x_{j-\frac{1}{2}} \le x \le x_{j+\frac{1}{2}})$$

$$x_{j\mp\frac{1}{2}} = (j \mp \frac{1}{2})\Delta x, \quad j = 0, \mp 1, \ldots.$$

The numerical flux function $h_f(a,b)$ is a Lipschitz continuous function of (a,b) for any fixed f, and is a continuous map from the space of real valued continuous functions defined on some real interval, into the real numbers, for fixed (a,b).

The most general class of schemes of this type whose solutions are known to converge to the correct physical solution for general $f(u)$ are E schemes. These were introduced and analyzed by the second author in a semi-discrete setting in [11], and then in the fully discrete setting by Tadmor in [15]. See [12] for convergence results for higher order accurate schemes approximating (1.1) when $f(u)$ is convex. (E schemes are at most first order accurate.)

An E scheme has a flux satisfying

$$\text{(2.2)} \quad \operatorname{sgn}(u_R - u_L)(h_f(u_R, u_L) - f(u)) \leq 0$$

for any $u \in I_{RL}$.

A special E scheme is due to Godunov [4] - its numerical flux is canonical in this class. This means

$$\text{(2.3)} \quad h_f^G(u_R, u_L) = \operatorname{sgn}(u_R - u_L) \min_{u \in I_{RL}} [\operatorname{sgn}(u_R - u_L) f(u)] = f(u(\zeta)), \quad \zeta = 0\,,$$

where $u(\zeta)$ is the entropy solution to (1.1).

Thus h_f is an E flux iff

$$\text{(2.4)} \qquad \operatorname{sgn}(u_R - u_L)\,(h_f(u_R, u_L) - h_f^G(u_R, u_L)) \leq 0$$

We may write any 3 point flux as

$$\text{(2.5)} \qquad h_f(u_R, u_L) = \frac{1}{2}\,[f(u_R) + f(u_L)] - \frac{1}{2\lambda}\,Q_{RL}(u_R - u_L),$$

defining the numerical viscosity Q_{RL}, which we take to be nonnegative.

A scheme has an E flux iff

(2.6) $$Q_{RL} \geq Q_{RL}^G$$

E schemes converge if their viscosities satisfy an upper bound.

(2.7) $$Q_{RL} \leq \frac{1}{2},$$

and a CFL condition.

(2.8) $$\lambda \sup|f'(u)| \leq \frac{1}{2},$$

where the sup is taken over the convex hull of the initial data, and $\lambda = (\Delta t/\Delta x)$.

Godunov's scheme is obtained by solving two noninteracting Riemann problems, equation (1.1), with initial data:

$$u(x,0) = u_{j-1}^n, \quad x \leq x_{j-\frac{1}{2}}$$

$$u(x,0) = u_j^n, \quad x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}$$

$$u(x,0) = u_{j+1}^n, \quad x_{j+\frac{1}{2}} \leq x.$$

We obtain $\tilde{u}_j(x,t)$, and compute at $t = \Delta t = \lambda \Delta x$, then average (for $|\lambda\, f'(u)| \leq \frac{1}{2}$):

$$\frac{1}{\Delta x} \int_{I_j} \tilde{u},(x,\Delta t)\, dx = u_j^{n+1} = u_j^n - \lambda(h^G(u_{j+1}^n, u_j^n) - h^G(u_j^n, u_{j-1}^n)).$$

Thus, Godunov's flux, as defined by (2.3), on a grid moving with speed $x = \zeta t$, is

$$h_f^G(u_R,u_L) = f(u(\zeta)) - \zeta u(\zeta),$$

hence, by formula (1.16) in [11], it follows that

$$u(\zeta) = -\frac{d}{d\zeta} h_{f_\zeta}^G(u_R,u_L). \tag{2.9}$$

Since $u(\zeta)$ is monotone, it follows that

$$\mathrm{sgn}(u_R - u_L) \frac{d^2}{d\zeta^2} h_{f_\zeta}^G(u_R,u_L) \le 0. \tag{2.10}$$

Thus, we shall consider the class of the 2 point (not necessarily E) fluxes, which, for fixed u_R, u_L, are convex (concave) functions of ζ in the sense described above. They must satisfy two conditions:

C(1) $\quad h_{f_\zeta}(u_R,u_L) \equiv f(u_L) - \zeta u_L, \quad \text{if } \zeta \le \zeta_L \le \min\limits_{I_R L} f'(u)$

$\quad\quad\; h_{f_\zeta}(u_R,u_L) \equiv f(u_R) - \zeta u_R, \quad \text{if } \zeta \ge \zeta_R \ge \max\limits_{I_R L} f'(u)$

C(2) $\quad \mathrm{sgn}(u_R - u_L) \dfrac{d^2}{d\zeta^2} h_{f_\zeta}(u_R,u_L) \le 0, \quad \text{for } \zeta_L \le \zeta \le \zeta_R.$

We next define a monotone function for any such $h_f(u_R,u_L)$ via a generalization of (2.9)

$$z(\zeta) = -\frac{d}{d\zeta} h_{f_\zeta}(u_R,u_L). \tag{2.11}$$

Conditions C guarantee that $z(\zeta)$ is an approximate Riemann solver for (1.1). The corresponding flux function, $\hat{f}(u)$, is obtained via the following:

THEOREM (2.1). The approximate Riemann solver, $z(\zeta)$, is the solution to the

Riemann problem.

$$z_t + \hat{f}_h(z)_x = 0$$

$$z(x,0) \equiv u_L, \quad x \le 0$$

$$z(x,0) \equiv u_R, \quad x > 0$$

where

$$\text{(2.12)} \quad \hat{f}_h(u) = \max_{\zeta_L \le \zeta \le \zeta_R} [\zeta u + h_{f_\zeta}(u_R,u_L)], \quad \text{if } u_L \le u_R$$

$$\hat{f}_h(u) = \min_{\zeta_L \le \zeta \le \zeta_R} [\zeta u + h_{f_\zeta}(u_R,u_L)], \quad \text{if } u_L > u_R.$$

PROOF. This is a direct consequence of Theorem (1.2), and conditions C.

Next, we may combine Theorems (1.3) and (2.1) to obtain:

THEOREM (2.2). The approximate Riemann solver, $z(\zeta)$, *is consistent with all entropy inequalities, iff* $h_{f_\zeta}(u_R,u_L)$ *is an E scheme for all* ζ, $\zeta_L \le \zeta \le \zeta_R$ *, i.e., iff*

$$\text{sgn}(u_R - u_L)\, h_{f_\zeta}(u_R,u_L) \le \text{sgn}(h_R - u_L)(f(u) - \zeta u)$$

for all u *between* u_L *and* u_R *and between* ζ_L *and* ζ_R.

It is consistent with a single entropy inequality (where the entropy function V depends on u_L, u_R, f and h) iff there exists some $u_0 \in I_{RL}$, such that

$$\text{sgn}(u_R - u_L)\, h_{f_\zeta}(u_R,u_L) \le \text{sgn}(u_R - u_L)(f(u_0) - \zeta u_0)$$

for all ζ, $\zeta_L \le \zeta \le \zeta_R$.

PROOF. By inequality (1.17) and Theorem (2.1), it follows that inequality

$$\operatorname{sgn}(u_R - u_L)[\zeta h + h_{f_\zeta}(u_R,u_L)] \le \operatorname{sgn}(u_R - u_L)f(u),$$

for all the appropriate u and ζ, is valid iff z is consistent with all entropy inequalities. The first result is immediate.

Inequality (1.16) is valid for some convex $V \in C^2$ iff there exists some u_0 in the interval for which

$$\operatorname{sgn}(u_R - u_L)\hat{f}_h(u_0) \le \operatorname{sgn}(u_R - u_L)f(u_0)$$

By Theorem (2.1), this is valid, iff

$$\operatorname{sgn}[u_R - u_L][\zeta u_0 + h_{f_\zeta}(u_R,u_L)] \le \operatorname{sgn}(u_R - u_L)f(u_0),$$

for all $\zeta \in [\zeta_L,\zeta_R]$.

The second result is immediate.

Remark (2.1). If, in (1.1), we make a change of variables

$$x \to x - \zeta t$$
$$t \to t$$

then (1.1) is replaced by

(2.11)(a) $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u) - \zeta u) = 0, \quad t > 0, -\infty < x < \infty$$

(2.11)(b) $$u(x,0) \equiv u_L, \quad x < 0$$

$$u(x,0) \equiv u_R, \quad x > 0.$$

Thus we can view $h_{f_\zeta}(u_R, u_L)$ as the numerical flux along the ray $x = \zeta t$, i.e. on a grid moving with speed ζ. Thus $z(\zeta)$ is the corresponding value of the approximate Riemann solver on this ray.

Remark (2.2). Any numerical flux satisfying C(1), C(2), is exactly Godunov's flux for $\hat{f}_h$ evaluated on a moving grid.

III. *Examples of Numerical Fluxes, Transforms, and Approximate Riemann Solvers*

We begin with the canonical example:

Example (3.1): Godunov's scheme [4]

As discussed in section II, the flux is defined by

$$(3.1) \qquad h_f^G(u_R,u_L) = \operatorname{sgn}(u_R - u_L)\min_{u \in I_{RL}} \operatorname{sgn}(u_R - u_L)\, f(u) = f(u(\zeta))|_{\zeta=0},$$

where $u(\zeta)$ is the entropy solution to (1.1).

Then for $u \in I_{RL}$

$$(3.2)(a) \qquad \hat{f}_{h_f^G}(u) = \text{convex hull of } f(u), \quad \text{if } u_L < u_R$$

$$(3.2)(b) \qquad \hat{f}_{h_f^G}(u) = \text{concave hull of } f(u), \quad \text{if } u_L > u_R.$$

This is merely a restatement of Lax's well known result: the solution to (1.1) is the same as would be obtained by replacing $f(u)$ by its convex (concave) hull, between u_L and u_R [8].

Example (3.2): Engquist-Osher scheme [2]

$$(3.3) \qquad h_f^{EO}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{1}{2}\int_{u_L}^{u_R} |(f'(s)|\, ds$$

$$(3.4) \quad h_{f_\zeta}^{EO}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\zeta}{2}(u_L + u_R) - \frac{1}{2}\int_{u_L}^{u_R} |(f'(s) - \zeta)|\, ds.$$

It is clear that condition C(1) is valid for

$\zeta_L = \min_I f'(u), \quad \zeta_R = \max_I f'(u).$

Also

$$(3.5) \qquad z(\zeta) = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\int_{u_L}^{u_R} \mathrm{sgn}(\zeta - f'(s))\, ds.$$

Since $\mathrm{sgn}(\zeta - f'(s))$ is a non-decreasing function of ζ, hypotheses C(2) is valid.

The transformed function, for $u_L < u_R$, satisfies

(3.6)(a) $\hat{f}_{h_f^{EO}}(u) = f(u)$ if f *is* convex

(3.6)(b) $\hat{f}_{h_f^{EO}}(u) = f(u_L) + f(u_R) - f(u_L + u_R - u)$, if f *is* concave.

This is the mirror image of the graph of $f(u)$ in the chord connecting u_L to u_R. The chord is itself the graph of $\hat{f}_{h_f^G}(u)$.

Example 3.3. Roe's (Murman's) scheme [13] (Not an E scheme.)

$$(3.7) \qquad h_f^R(u_R,u_L) = \frac{1}{2}(f(u_R) + f(u_L)) - \frac{1}{2}|s|(u_R - u_L)$$

where

$$(3.8) \qquad s = \frac{f(u_R) - f(u_L)}{u_R - u_L}.$$

Thus

$$(3.9) \quad h_{f_\zeta}^R(u_R,u_L) = \frac{1}{2}(f(u_R) + f(u_L)) - \frac{\zeta}{2}(u_L + u_R) - \frac{1}{2}|s - \zeta|(u_L - u_R)$$

$$= f(u_L) - \zeta u_L, \text{ if } \zeta \leq s$$

$$= f(u_R) - \zeta u_R, \text{ if } \zeta > s.$$

Hypotheses C(1) are clearly valid for any ζ_L, ζ_R

Also

$$z(\zeta) = u_L, \zeta \le s \tag{3.10}$$

$$z(\zeta) = u_R, \zeta > s$$

and

$$\hat{f}_{h_f^R}(u) = f(u_L) + (u - u_L)\, s, \quad \text{for } u \in I_{RL}, \tag{3.11}$$

i.e., the linear function connecting $(u_L, f(u_L))$ to $(u_R, f(u_R))$.

By Theorem (1.3) and (3.11), the approximate Riemann solver is consistent with all entropy inequalities iff the chord connecting u_L to u_R lies below (above) the graph of $f(u)$ if $u_L < u_R$ $(u_L > u_R)$.

Various obvious entropy "fixes" of Roe's scheme and for the extension to systems exist - see e.g. [4].

Example (3.4). Lax-Friedrichs' Scheme.

$$h_f^{LF}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{1}{2\lambda}\,Q(u_R - u_L) \tag{3.12}$$

for $0 < Q \le \frac{1}{2}$, Q constant.

Thus

$$h_{f_\zeta}^{LF}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\zeta}{2}\,(u_R + u_L) - \frac{1}{2\lambda}\,Q(u_R - u_L) \tag{3.13}$$

for

$$\text{(3.14)} \qquad \zeta_L = s - \frac{Q}{\lambda} \le \zeta \le s + \frac{Q}{\lambda} = \zeta_R .$$

Hyothesis C(2) is obviously valid.

Thus, we have:

$$\text{(3.15)} \qquad z(\zeta) \equiv u_L, \quad \zeta \le \zeta_L$$

$$z(\zeta) = \frac{1}{2}(u_R + u_L) = u_m, \quad \zeta_L \le \zeta \le \zeta_R$$

$$z(\zeta) = u_R, \quad \zeta \ge \zeta_R .$$

and

(3.16)(a) if $u_L \le u_R$

$$\hat{f}_{h_f^{LF}}(u) = \left(s - \frac{Q}{\lambda}\right)(u - u_m) + \frac{1}{2}(f(u_R) + f(u_L)) - \frac{1}{2\lambda} Q(u_R - u_L) \quad \text{for } u \le u_m .$$

$$= \left(s + \frac{Q}{\lambda}\right)(u - u_m) + \frac{1}{2}(f(u_R) + f(u_L)) - \frac{1}{2\lambda} Q(u_R - u_L) \text{ for } u \ge u_m .$$

(3.16)(b) if $u_L > u_R$

$$\hat{f}_{h_f^{LF}}(u) = \left(s + \frac{Q}{\lambda}\right)(u - u_m) + \frac{1}{2}(f(u_R) + f(u_L)) - \frac{1}{2\lambda} Q(u_R - u_L) \quad \text{for } u \ge u_m .$$

$$= \left(s - \frac{Q}{\lambda}\right)(u - u_m) + \frac{1}{2}(f(u_R) + f(u_L)) - \frac{1}{2\lambda} Q(u_R - u_L) \quad \text{for } u \le u_m .$$

Thus $\hat{f}_{H_f^F}(u)$ is a piecewise linear function whose graph lies below (above) that of $f(u)$ if $u_L < u_R$, ($u_L > u_R$), if the CFL restriction:

$$\lambda(s + \max_{I_{RL}} |f'(u)|) \le Q, \tag{3.17}$$

is valid, in which case hypothesis C(1) is satisfied.

Example (3.5) Lax-Wendroff scheme [9], (second order accurate, without entropy fix).

$$h_f^{LW}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\lambda}{2}\,a_m\,s(u_R - u_L) \tag{3.18}$$

where

$$a_m = f'(u_m)$$

Thus

$$h_{f_\zeta}^{LW}(u_R,u_L) = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\zeta}{2}\,(u_R + u_L). \tag{3.19}$$

$$- \frac{\lambda}{2}(a_m - \zeta)(s - \zeta)(u_R - u_L).$$

We see for $\zeta = s$, that the right side of (3.19) is

$$f(u_R) - s\,u_R = f(u_L) - su_L,$$

Letting $\zeta = a_m - 1/\lambda$, also gives us the value: $f(u_R) - (a_m - 1/\lambda)\,u_R$. Thus the only way to have hypothesis C(1) be valid is by letting

$$h_{f_\zeta}^{LW}(u_R,u_L) \equiv h_{f_\zeta}^{R}(u_R,u_L)$$

Both these schemes have the same unfortunate behavior at expansion shocks moving with speed ζ, as was seen in [2], [10].

Example (3.6). Lax-Wendroff scheme modified with entropy fix [10].

$$(3.20)\quad h_f^{LWE} = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\lambda}{2}\,s^2(u_R - u_L)\,,$$

$$- \frac{C}{2}\,|\,f'(u_R) - f'(u_L)|(u_R - u_L)\,,$$

where $C > 0$, will be chosen below.

Thus

$$(3.21)\quad h_{f_\zeta}^{LWE} = \frac{1}{2}\,(f(u_R) + f(u_L)) - \frac{\zeta}{2}(u_R + u_L)$$

$$- \frac{\lambda}{2}(s - \zeta)^2(u_R - u_L) - \frac{C}{2}\,|(f'(u_R) - f'(u_L)|(u_R - u_L).$$

A simple calculation gives us

$$(3.22)(a)\qquad \zeta_L^{(1)} = s + \frac{-1 + (1 - 4C\lambda\,|\,\Delta_+ f'|)^{\frac{1}{2}}}{2\lambda}$$

$$(3.22)(b)\qquad \zeta_R^{(1)} = s + \frac{1 - (1-4Cu\,|\Delta_+ f'|)^{\frac{1}{2}}}{2\lambda},$$

where we defined: $\Delta_+ f' = f'(u_R) - f'(u_L)$.

It follows that:

$$(3.23)\quad h_{f_{\zeta_R^{(1)}}} = f(u_L) - \zeta_L^{(1)} u_L.$$

$$h_{f_{\zeta_R^{(1)}}} = f(u_R) - \zeta_R^{(1)} u_R$$

under the restriction:

$$4C\lambda|\Delta_+ f'| < 1.$$

To have $\zeta_L^{(1)} = \zeta_L, \quad \zeta_R^{(1)} = \zeta_R,$ we need

$$s + \max_{I_{RL}} |f'| \le C\ |\Delta_+ f'| \tag{3.25}$$

In general, this restriction forces the scheme to be only first order accurate. Thus we omit it and define $h_{f_\zeta}^{LWE}$ as in (3.21) for $\zeta_L^{(1)} \le \zeta \le \zeta_R^{(1)}$, and set

$$h_{f_\zeta}^{LWE} = f(u_L) - \zeta u_L, \quad \zeta \le \zeta_L^{(1)} \tag{3.26}$$

$$h_{f_\zeta}^{LWE} = f(u_R) - \zeta u_R, \quad \zeta \ge \zeta_R^{(1)}.$$

The resulting $z(\zeta)$ is defined to be

$$z(\zeta) \equiv u_L, \quad \zeta \le \zeta_L^{(1)} \tag{3.27}$$

$$z(\zeta) = u_m + \lambda(\zeta - s)(u_R - u_L), \quad \zeta_L^{(1)} < \zeta < \zeta_R^{(1)}$$

$$z(\zeta) = u_R, \quad \zeta_R^{(1)} \le \zeta$$

By (3.22) and (3.25), this function is monotone, thus condition C(2) is valid. The associated transform is defined via

(3.28) for $(u_L < u_R)$

$$\hat{f}_{H_f^{LWE}}(u) = \zeta_L^{(1)}(u - u_L) + f(u_L), \text{ if } u_L \le u \le (\zeta_L^{(1)} - s)\lambda(u_R - u_L) + u_m.$$

$$\hat{f}_{H_f^{LWE}}(u) = \frac{(u - u_m)^2}{2\lambda(u_R - u_L)} + \frac{1}{2}(f(u_R) + f(u_L)) - \frac{C}{2} \mid \Delta_+ f' \mid (u_R - u_L)$$

$$+ s(u - u_m), \text{ if } (\zeta_L^{(1)} - s)\lambda(u_R - u_m) + u_m \le u \le (\zeta_R^{(1)} - s)\lambda(u_R - u_m) + u_m.$$

$$\hat{f}_{H_f^{LWE}}(u) = \zeta_R^{(1)}(u - u_R) + f(u_R), \text{ if } (\zeta_R^{(1)} - s)\lambda(u_R - u_L) + u_m \le u \le u_R.$$

The flux function is defined analogously for $u_L > u_m$.

Although Theorem (2.1) guarantees that this scheme cannot possibly satisfy all entropy inequalities and stay second order accurate, a single inequality follows easly under mild restrictions. For example, if $u_L < u_R$, we might take

$$\frac{1}{2}(f(u_R) + f(u_L)) - \frac{C}{2}|\Delta_+ f'|(u_R - u_L) \le f(u_m).$$

or

$$\frac{f(u_R) + f(u_L) - 2f(u_m)}{(u_R - u_L) \mid \Delta_+ f' \mid} \le C. \tag{3.29}$$

The quantity on the left above is bounded above by a positive number as $u_R \to u_L$ so (3.29) is possible for a fixed constant independent of $\mid u_R - u_L \mid$.

For a given fixed entropy, say $V(u) = \frac{1}{2}u^2$, the quantity C can be chosen large enough so that inequality (1.16) is valid. See [10] for related estimates.

IV. *Comparison with Other Approaches*

In [6], the authors construct a monotone, piecewise constant, approximate Riemann solver for (1.1) as follows:

$$z(\zeta) \equiv u_L, \quad \infty < \zeta \le \zeta_L^{(1)} \tag{4.1}$$

$$z(\zeta) \equiv \bar{u}, \quad \zeta_L^{(1)} < \zeta < \zeta_R^{(1)}$$

$$z(\zeta) \equiv u_R, \quad \zeta_R^{(1)} \le \zeta < \infty.$$

Consistency with conservation means

$$\zeta_L^{(1)}(u_L - \bar{u}) + \zeta_R^{(1)}(\bar{u} - u_R) + f(u_R) - f(u_L) = 0. \tag{4.2}$$

The associated flux function is piecewise linear, and defined via:

$$\hat{f}(u) = f(u_L) + \zeta_L^{(1)}(u - u_L), \quad \text{for } |u - u_L| \le |\bar{u} - u_L| \tag{4.3}$$

$$\hat{f}(u) = f(u_R) + \zeta_R^{(1)}(u - u_R), \quad \text{for } |u - u_R| \le |\bar{u} - u_R| .$$

By Theorem (1.3), a necessary and sufficient condition that $z(\zeta)$ be consistent with all entropy inequality is that the graph of $\hat{f}(u)$ be below (above) that of $f(u)$ if $u_L < u_R$ ($u_L > u_R$). This is implied by the restrictions imposed in [6].

If u_L is connected to u_R via a single entropy condition satisfying shock, we may take $\zeta_L^{(1)} = \zeta_R^{(1)} = s$, i.e., we have Roe's transform function.

If we take $\zeta_L^{(1)} = s - Q/\lambda$, $\zeta_R^{(1)} = s + Q/\lambda$, under hypothesis (3.19) we see that the Lax-Friedrichs flux transforms to $\hat{f}(u)$, with $\bar{u} = \frac{1}{2}(u_L + u_R) = u_m$.

In [5] the authors consider an approximate solution of (1.1) which is not a similarity solution. It could be modified slightly keeping its essential properties so that in this scalar case it becomes

$$z(\zeta) \equiv u_L, \quad -\infty < \zeta \leq s - \delta \tag{4.6}$$

$$z(\zeta) = u_L^*, \quad s - \delta < \zeta \leq s$$

$$z(\zeta) = u_R^*, \quad s \leq \zeta \leq s + \delta$$

$$z(\zeta) \equiv u_R, \quad s + \delta \leq \zeta < \infty ,$$

where $\delta > 0$, will be chosen below.

We futher restrict z to be monotone, which was not explicitly done in [5]. For this function to be consistent with conservation form means that:

$$u_L(s - \delta - \zeta_L) + u_L^* \delta + u_R^* \delta + u_R (\zeta_R - s - \delta)$$

$$= \zeta_R u_R - \zeta_L u_L - (f(u_R) - f(u_L))$$

or

$$(u_L^* - u_L) + (u_R^* - u_R) = 0 . \tag{4.7}$$

Harten and Lax compute

$$u_L^* = u_L - \lambda(g_{LR} - g(u_L)) .$$

$$u_R^* = u_R - \lambda(g(u_R) - g_{LR})$$

Here, in our language:

(4.8)(a) $$g(u) = f(u) - su = f_s(u).$$

and

(4.8)(b) $$g_{LR} = h_{f_s}(u_R, u_L),$$

for some numerical flux h_f.

Equality (4.7) is automatically valid in this case. In order that $z(\zeta)$ be monotone, it is necessary and sufficient that

(4.9)(a) $$\operatorname{sgn}(u_R - u_L)(h_{f_s}(u_R, u_L) - f_s(u_L)) \leq 0$$

(4.9)(b) $$\operatorname{sgn}(u_R - u_L)(h_{f_s}(u_R, u_L) - f_s(u_R)) \leq 0$$

(4.9)(c) $$|u_R - u_L| \geq \lambda \operatorname{sgn}(u_R - u_L) [f_s(u_R) + f_s(u_L) - 2h_{f_s}(u_R, u_L)] .$$

The inequalities (4.9)(a) and (4.9)(b) are valid iff the associated 3 point scheme is TVD, hence only first order accurate - see e.g., [15]. Inequality (c) is merely a CFL restriction.

The associated function $\hat{f}(u)$, is, of course, again piecewise linear. Its graph connects the four nodes $(u_L, f(u_L))$, $(u_L^*, f(u_L) + (s - \delta)(u_L^* - u_L))$, $(u_R^*, f(u_L) + s(u_R^* - u_L^*) + (s - \delta)(u_L^* - u_L)$, $(u_R, f(u_R))$.

The slopes of these three lines are $s - \delta$, s, $s + \delta$ respectively.

Again the graph of this function lies below (above) that of $f(u)$ on I_{RL} for $u_L < u_R$, $(u_L > u_R)$ iff all entropy inequalities are satisfied.

Harten and Lax define h_{f_ζ} such that

$$u_L^* = u_L + \lambda\beta(u_R - u_L), \quad u_R^* = u_R - \lambda\beta(u_R - u_L);$$

thus given $\lambda\beta \le \frac{1}{2}$, $z(\zeta)$ is a monotone function. The inequalities which are necessary and sufficient for consistency with all entropy inequalities are:

$$\text{(4.10)(a)} \qquad \delta\lambda\beta \ge -\min_{u \in I_{RL}} \frac{(f_s(u) - f_s(u_L))}{u_R - u_L} \ge 0$$

$$\text{(4.10)(b)} \qquad 1 \ge \max_{u \in I_{RL}} \frac{f_s(u_R) - f_s(u)}{\delta(u_R - u)}$$

$$\text{(4.10)(c)} \qquad 1 \ge \max_{u \in I_{RL}} \frac{f_s(u_L) - f_s(u)}{\delta(u - u_L)} \ .$$

These inequalities are all compatible. If we take, for example, $\delta = 1/\lambda$, the last two become CFL conditions, and the first is an improvement over the corresponding estimate in [5].

BIBLIOGRAPHY

[1] Y. Brenier, "Total variation estimate in Glimm's method for large Courant numbers," Preprint.

[2] B. Engquist and S. Osher, Stable and entropy condition satisfying approximations for transonic flow calculations, Math. Comp., **34** (1980), 45-75.

[3] J. Glimm, "Solutions in the large for nonlinear hyperbolic systems of equations," Comm. Pure Appl. Math, **18** (1965), 697-715.

[4] S.K. Godunov, "A finite difference method for the numerical solution of discontinuous solutions of the equations of fluid dynamics," Mat. Sb., **47** (1959), 271-290.

[5] A. Harten and P.D. Lax, "A random choice finite difference scheme for hyperbolic conservation laws," SINUM, **18** (1981), 289-315.

[6] A. Harten, P.D. Lax and B. van Leer, "On upstream differencing and Godunov-type schemes for hyperbolic conservation laws, SIAM Review, **25** (1983), 35-61.

[7] S.N. Kruzkov, First order quasi-linear equations in several independent variables, Math. USSR Sb., **10** (1970), 217-243.

[8] P.D. Lax, "Hyperbolic systems of conservation laws and the mathematical theory of shock waves," CBMS Regional Conference Series in Applied Mathematics, SIAM, (1972).

[9] P.D. Lax and B. Wendroff, "Systems of conservation laws," Comm. Pure Appl. Math., **23** (1960), 217-237.

[10] A. Majda and S. Osher, Numerical viscosity and the entropy condition, Comm. Pure Appl. Math., **32** (1979), 797-838.

[11] S. Osher, Riemann solvers, the entropy condition, and difference approximations, SINUM, **21** (1984), 217-235.

[12] S. Osher and E. Tadmor, "On the convergence of difference approximations to scalar conservation laws," in preparation.

[13] P.L. Roe, Approximate Riemann solvers, parameter vectors, and difference schemes, J. Comp. Phys., **43** (1981), 357-372.

[14] P.L. Roe, Some contributions to the modelling of discontinuous flows, to appear in Proceedings of AMS-SIAM 1983 Summer Seminar, La Jolla, CA., eds. B. Engquist, S. Osher, R.C.J. Somerville.

[15] E. Tadmor, Numerical viscosity and the entropy condition for conservative difference schemes, NASA Contractor Report 172141, (1983), NASA Langley, Math Comp. (to appear).